EOS RESEARCH PROJECT: synopsis

"High-dimensional expanders and Kac–Moody–Steinberg groups"

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State of the art

The study of *expander graphs* originates from computer science and can be traced back to Pinsker in 1973 [6]. An expander graph is, roughly speaking, a sparse but nevertheless sufficiently connected graph, making it useful for constructing computer networks that are reliable as well as cost-effective. A huge area of research has emerged from this, involving both mathematics and computer science (see [4] and references therein). Two excellent surveys are [2, 4].

More recently, the notion of expander graphs has been extended to higher dimensions, giving rise to (families of) simplicial complexes called *high-dimensional expanders*. Their study provides the main motivation for this research proposal. We refer to the excellent survey [5].

A recurring theme of many of the known constructions of high-dimensional expanders is the use of *Bruhat–Tits buildings* of algebraic groups over local fields, the expanders being obtained as suitable quotients of such buildings by cocompact arithmetic lattices.

Our ambition with this project is to contribute to the state-of-the-art by providing new abundant sources of examples arising from simplicial complexes associated with a non-classical family of groups, called *Kac–Moody–Steinberg groups* (*KMS groups* for short).

Scientific research hypothesis and objectives

KMS groups provide a remarkable but little studied class of finitely presented groups which is interesting in its own right. They are defined as fundamental groups of certain complexes of finite p-groups, where p is a fixed prime number. The finite p-groups in question are the positive unipotent subgroups of basic Levi subgroups of spherical type in a 2-spherical *Kac–Moody group* over a finite field. One example of a KMS group is the group $\mathcal{U}$ with presentation

$$\mathcal{U} = \langle x_1, \dots, x_d \mid x_i^p,\ [[x_i, x_j], x_j]\ (i \neq j) \rangle.$$

In this project, we make the hypothesis that, by unveiling the combinatorial, geometric, Lie-algebraic and cohomological properties of KMS groups, we will be able to provide key examples relevant not only to the vibrant emerging theory of high-dimensional expanders, but also to various other core problems in geometric group theory concerning the residual properties of hyperbolic groups, lattice envelopes, cohomology vanishing and group stability. The starting point of this novel approach is inspired by the recent preliminary work in [3] and Section 7 of the preprint [1].

Regarding the theory of high-dimensional expanders, the main contribution of the project will be achieved by the following milestone.

Milestone. *Use KMS groups to construct families of high-dimensional expanders of* ***bounded degree*** *associated with finite simple Lie type groups of* ***unbounded rank*** *(resp. with alternating groups of* ***unbounded degree****).*

The milestone will be obtained as a byproduct of our systematic investigations of KMS groups, which are articulated around various precise research questions organized into 3 Work Packages: (WP1) Algebraic structure; (WP2) Geometric structure; (WP3) Lie-algebraic methods. The cross-sectional objectives of the proposal across these WPs can be summarized as follows:

- Study the global structure of KMS groups and the associated geometries.
- Study the local structure of KMS groups and the associated geometries.
- Study the algebraic quotients of KMS groups.
- Study the spectral and cohomological properties of KMS groups.

This project addresses cutting-edge research topics with a lot to be discovered, but at the same time these topics have already shown their potential. We believe that our investigations will have a large impact not only on the specific research questions that will be investigated in order to reach the Milestone, but that our contributions in this context will moreover be the starting point for future research projects in geometric group theory.

Research team

The success of this project requires the hiring of 3 PhD students (4 years each) and 4 post-docs (2 years each).

- One PhD student in Louvain-la-Neuve will investigate the algebraic structure of KMS groups (acylindric hyperbolicity, Kazhdan's property (T), …) and benefit from the expertise of the research group in Louvain-La-Neuve in geometric group theory and representation theory (cosupervision PEC–TM).
- One PhD student in Ghent will investigate geometric expansion properties of KMS groups, and benefit from the expertise of the research group in Ghent on incidence geometry (including the theory of buildings) and group theory (supervision TDM).
- One PhD student will investigate the interplay between the algebraic and Lie-theoretic aspects, and will benefit from the expertise of both research groups on Kac–Moody algebras, groups and Lie algebras (cosupervision TDM–TM). He/she will first spend two years in Louvain-la-Neuve followed by two years in Ghent (joint PhD).
- Two post-docs in Louvain-la-Neuve will investigate cohomological properties and unitary representations, and benefit from the expertise of the research group in Louvain-La-Neuve on representation theoretic aspects of KMS and Kac–Moody groups (cosupervision PEC–TM).
- Two post-docs in Ghent will work on geometric and Lie-theoretic aspects, and benefit from the expertise of the research group in Ghent on Lie algebras, non-associative algebras and group theory (supervision TDM).

Needless to say, this is only a rough sketch, and we will be open for new research opportunities along the way.

We expect a close interaction between all members involved in this project. We plan to organize monthly "sync events", 3-day workshops twice a year, and we will organize two larger conferences, one in the beginning of the project (planned in 2023) and one towards the end.

References

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